

Hölder conditions for endomorphisms of hyperbolic groups

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ABSTRACT

It is proved that an endomorphism φ of a hyperbolic group G satisfies a Hölder condition with respect to a visual metric if and only if φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G . If G is virtually free or torsion-free co-hopfian, then φ is uniformly continuous if and only if it satisfies a Hölder condition if and only if it is virtually injective. Lipschitz conditions are discussed for free group automorphisms.

1 Introduction

The concept of boundary of a free group has been for a number of years a major subject of research from geometric, topological, dynamical, algebraic or combinatorial viewpoints. The boundary of F_A , denoted by ∂F_A , can be defined as the set of all infinite reduced words on $\tilde{A} = A \cup A^{-1}$, but the topological (metric) structure is of utmost importance. It can be defined through the *prefix metric*. Given $u, v \in F_A$, let $u \wedge v$ denote the longest common prefix of u and v . An ultrametric $p_A : F_A \rightarrow F_A \rightarrow \mathbb{R}_0^+$ is defined by

$$p_A(u, v) = \begin{cases} 2^{-|u \wedge v|} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases}$$

The completion $(\widehat{F}_A, \widehat{p}_A)$ can be described as

$$\widehat{F}_A = F_A \cup \partial F_A,$$

and the metric \widehat{p}_A is nothing but the prefix metric defined for finite and infinite reduced words altogether.

The theory of hyperbolic groups generalizes many aspects of free groups, and we can endow the boundary of a hyperbolic group with a metric structure proceeding analogously. This can be achieved with the help of the Gromov product and the visual metrics $\sigma_{p,\gamma}^A$. If $G = F_A$, $p = 1$ and $\gamma = \ln 2$, then $\sigma_{p,\gamma}^A$ is precisely the prefix metric defined above.

The completion $(\widehat{G}, \widehat{\sigma}_{p,\gamma}^A)$ of $(G, \sigma_{p,\gamma}^A)$ produces the boundary $\partial G = \widehat{G} \setminus G$ and its metric structure, which induces the Gromov topology on ∂G . This same topology can be induced by any of the visual metrics $d \in V^A(p, \gamma, T)$. These are the metrics considered in this paper, and their extensions \widehat{d} to \widehat{G} .

Since the completion $(\widehat{G}, \widehat{d})$ is also compact, the endomorphisms of G which admit a continuous extension to the boundary are precisely the uniformly continuous ones. It is thus a natural problem to determine which endomorphisms of G admit such a continuous extension. It is well known that automorphisms do.

Uniform continuity is implied by a Hölder condition. A mapping $\varphi : (X, d) \rightarrow (X', d')$ satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(x\varphi, y\varphi) \leq K(d(x, y))^r$$

for all $x, y \in X$. A Hölder condition of exponent 1 is a *Lipschitz condition*. Using an analogy with notions from complexity theory in theoretical computer science (and inverting the exponent), we may identify Hölder conditions with *polynomial complexity* and Lipschitz conditions with *linear complexity*.

In this paper, we are interested mainly on Hölder conditions for endomorphisms, with respect to visual metrics. Given the exponential in the definition of the visual metric, it is not surprising that this reduces to some Lipschitz type condition involving Gromov products. As a preliminary result, we show that all visual metrics on a hyperbolic group are Hölder equivalent.

In the main theorem of the paper (Theorem 4.3), we establish several equivalent conditions for a nontrivial endomorphism of a hyperbolic group to satisfy a Hölder condition. The most interesting is undoubtedly the last one: p must be virtually injective and $G\varphi$ must be a quasiconvex subgroup of G . This second requirement may be removed if the group is virtually free or torsion-free cohopfian, when we also show that all uniformly continuous endomorphisms satisfy a Hölder condition. The second author had proved in [16, Proposition 7.2] that a nontrivial endomorphism of a finitely generated virtually free group is uniformly continuous if and only if it is virtually injective. We ignore whether this is also true for hyperbolic groups.

We discuss also Lipschitz conditions for automorphisms. It is easy to see that every inner automorphism of a hyperbolic group satisfies a Lipschitz condition, but we have only succeeded on finding a precise characterization in the free group case. With respect to the canonical basis, Lipschitz conditions occur only for compositions of permutation automorphisms with inner automorphisms. If we allow arbitrary finite generating sets, we have only the inner automorphisms, and the same happens if we allow arbitrary bases in rank ≥ 3 . In rank 2, we obtain an intermediate class of automorphisms.

One of the motivations for our work is the possibility of defining new pseudometrics on $\text{Aut}(G)$ for every hyperbolic group G . Given a virtually injective endomorphism φ of G and a visual metric d on G , write

$$\|\varphi\|_d = \ln(\inf\{r \geq 1 \mid \varphi \text{ satisfies a Hölder condition of exponent } r^{-1} \text{ with respect to } d\}).$$

Since

$$\|\varphi\psi\|_d \leq \|\varphi\|_d + \|\psi\|_d \tag{1}$$

for all virtually injective endomorphisms φ, ψ of G , we call $\|\cdot\|_d$ a seminorm. All inner automorphisms have seminorm 0.

Now we define a pseudometric \overline{d} on $\text{Aut}(G)$ by

$$\overline{d}(\varphi, \psi) = \max\{\|\varphi^{-1}\psi\|_d, \|\psi^{-1}\varphi\|_d\}.$$

The inequality (1) implies the triangular inequality for \overline{d} . This pseudometric is the object of ongoing work by the authors.

The paper is organized as follows. In Section 2 we present basic concepts and notation for hyperbolic groups. Visual metrics and some of their basic properties in connection with Hölder conditions are discussed in Section 3. The main results of the paper, characterizing which uniformly continuous endomorphisms satisfy a Hölder condition, are presented in Section 4. Simplifications for the case of virtually free or torsion-free co-hopfian hyperbolic groups are discussed in Section 5. In Section 6, we discuss Lipschitz conditions. Finally, some open problems are proposed in Section 7.

2 Hyperbolic groups

We present in this section well-known facts regarding hyperbolic spaces and hyperbolic groups. The reader is referred to [2, 5] for details.

A mapping $\varphi : (X, d) \rightarrow (X', d')$ between metric spaces is called an *isometric embedding* if $d'(x\varphi, y\varphi) = d(x, y)$ for all $x, y \in X$. A surjective isometric embedding is an *isometry*.

A metric space (X, d) is said to be *geodesic* if, for all $x, y \in X$, there exists an isometric embedding $\xi : [0, s] \rightarrow X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of \mathbb{R} . We call ξ a *geodesic* of (X, d) . We shall often call $\text{Im}(\xi)$ a geodesic as well. In this second sense, we may use the notation $[x, y]$ to denote an arbitrary geodesic connecting x to y . Note that a geodesic metric space is always (path) connected.

A *quasi-isometric embedding* of metric spaces is a mapping $\varphi : (X, d) \rightarrow (X', d')$ such that there exist constants $\lambda \geq 1$ and $K \geq 0$ satisfying

$$\frac{1}{\lambda}d(x, y) - K \leq d'(x\varphi, y\varphi) \leq \lambda d(x, y) + K$$

for all $x, y \in X$. We may call it a (λ, K) -quasi-isometric embedding if we want to stress the constants. If in addition

$$\forall x' \in X' \exists x \in X : d'(x', x\varphi) \leq K,$$

we say that φ is a *quasi-isometry*.

Two metric spaces (X, d) and (X', d') are said to be *quasi-isometric* if there exists a quasi-isometry $\varphi : (X, d) \rightarrow (X', d')$. Quasi-isometry turns out to be an equivalence relation on the class of metric spaces. A *quasi-geodesic* of (X, d) is a quasi-isometric embedding $\xi : [0, s] \rightarrow X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of \mathbb{R} .

Let (X, d) be a geodesic metric space. Given $x_0, x_1, x_2 \in X$, a *geodesic triangle* $[[x_0, x_1, x_2]]$ is a collection of three geodesics $[x_0, x_1]$, $[x_1, x_2]$ and $[x_2, x_0]$ in X .

Given $\delta \geq 0$, we say that (X, d) is δ -*hyperbolic* if

$$\forall y \in [x_0, x_2] \quad d(y, [x_0, x_1] \cup [x_1, x_2]) \leq \delta \tag{2}$$

holds for every geodesic triangle $[[x_0, x_1, x_2]]$ in X . If this happens for some $\delta \geq 0$, we say that (X, d) is *hyperbolic*.

Given $Y, Z \subseteq X$ nonempty, the *Hausdorff distance* between Y and Z is defined by

$$\text{Haus}(Y, Z) = \max\{\sup_{y \in Y} d(y, Z), \sup_{z \in Z} d(z, Y)\}.$$

If (X, d) is δ -hyperbolic and $\lambda \geq 1$, $K \geq 0$, it follows from [5, Theorem 5.4.21] that there exists a constant $R(\delta, \lambda, K)$, depending only on δ, λ, K , such that any geodesic and (λ, K) -quasi-geodesic

in X having the same initial and terminal points lie at Hausdorff distance $\leq R(\delta, \lambda, K)$ from each other. This constant will be used in the proof of several results.

Given a subset A of a group G , we denote by $\langle A \rangle$ the subgroup of G generated by A . We assume throughout the paper that generating sets are finite.

Given $G = \langle A \rangle$, we write $\tilde{A} = A \cup A^{-1}$. The *Cayley graph* $\Gamma_A(G)$ has vertex set G and edges of the form $g \xrightarrow{a} ga$ for all $g \in G$ and $a \in \tilde{A}$. The *geodesic metric* d_A on G is defined by taking $d_A(g, h)$ to be the length of the shortest path connecting g to h in $\Gamma_A(G)$.

Since $\text{Im}(d_A) \subseteq \mathbb{N}$, then (G, d_A) is not a geodesic metric space. However, we can remedy that by embedding (G, d_A) isometrically into the *geometric realization* $\bar{\Gamma}_A(G)$ of $\Gamma_A(G)$, when vertices become points and edges become segments of length 1 in some (euclidean) space, intersections being determined by adjacency only. With the obvious metric, $\bar{\Gamma}_A(G)$ is a geodesic metric space, and the geometric realization is unique up to isometry. We denote also by d_A the induced metric on $\bar{\Gamma}_A(G)$.

We say that the group G is *hyperbolic* if the geodesic metric space $(\bar{\Gamma}_A(G), d_A)$ is hyperbolic.

If A' is an alternative finite generating set of G and

$$N_{A,A'} = \max(\{d_{A'}(1, a) \mid a \in A\} \cup \{d_A(1, a') \mid a' \in A'\}), \quad (3)$$

it is immediate that

$$\frac{1}{N_{A,A'}} d_{A'}(g, h) \leq d_A(g, h) \leq N_{A,A'} d_{A'}(g, h) \quad (4)$$

holds for all $g, h \in G$, hence the identity mapping $(G, d_A) \rightarrow (G, d_{A'})$ is a quasi-isometry. It follows easily that the concept of hyperbolic group is independent from the finite generating set considered, but the hyperbolicity constant δ may vary with the generating set.

Condition (2), which became the most popular way of defining hyperbolic group, is known as *Rips condition*. An alternative approach is given by the concept of Gromov product, which we now define. It can be defined for every metric space.

Given $g, h, p \in G$, we define

$$(g|h)_p^A = \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)).$$

We say that $(g|h)_p^A$ is the *Gromov product* of g and h , taking p as basepoint.

The following result is well known:

Proposition 2.1 *The following conditions are equivalent for a group $G = \langle A \rangle$:*

- (i) G is hyperbolic;
- (ii) there exists some $\delta \geq 0$ such that

$$(g_0|g_2)_p^A \geq \min\{(g_0|g_1)_p^A, (g_1|g_2)_p^A\} - \delta \quad (5)$$

holds for all $g_0, g_1, g_2, p \in G$.

Let H be a subgroup of a hyperbolic group $G = \langle A \rangle$ and let $q \geq 0$. We say that H is *q-quasiconvex* with respect to A if

$$\forall x \in [h, h'] \quad d_A(x, H) \leq q$$

holds for every geodesic $[h, h']$ in $\bar{\Gamma}_A(G)$ with endpoints in H . We say that H is *quasiconvex* if it is q -quasiconvex for some $q \geq 0$. Like most other properties in the theory of hyperbolic groups, quasiconvex does not depend on the finite generating set considered [2, Section III.Γ.3].

A (finitely generated) subgroup of a hyperbolic group needs not be hyperbolic, but a quasiconvex subgroup of a hyperbolic group is always hyperbolic. The converse is not true in general. Quasiconvex subgroups occur quite frequently in the theory of hyperbolic groups. In fact, non quasi convex subgroups are a relatively rare phenomenon, see [8].

We present next a model for the boundary of G .

Given a mapping $\varphi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$, we write

$$\lim_{i,j \rightarrow +\infty} (i, j)\varphi = \lim_{n \rightarrow +\infty} (\inf\{(i, j)\varphi \mid i, j \geq n\}).$$

Fix a generating set A for G and $p \in G$. We say that a sequence $(g_n)_n$ on G is a *Gromov sequence* if

$$\lim_{i,j \rightarrow +\infty} (g_i | g_j)_p^A = +\infty.$$

This property is independent from both A and g . Two Gromov sequences $(g_n)_n$ and $(h_n)_n$ on G are *equivalent* if

$$\lim_{n \rightarrow +\infty} (g_n | h_n)_p^A = +\infty.$$

We denote by $[(g_n)_n]$ the equivalence class of the Gromov sequence $(g_n)_n$. The set of all such equivalence classes is one of the standard models for the boundary ∂G , and is adopted in this paper.

We can identify G with the set of all constant sequences $(g)_n$ on G , and consider

$$\widehat{G} = \partial G \cup \{(g)_n \mid g \in G\}.$$

The Gromov product is extended to \widehat{G} by setting

$$(\alpha | \beta)_p^A = \sup\{ \lim_{i,j \rightarrow +\infty} (g_i | h_j)_p^A \mid (g_n)_n \in \alpha, (h_n)_n \in \beta \}$$

for all $\alpha, \beta \in \widehat{G}$.

3 The visual metrics

Let $G = \langle A \rangle$ be a hyperbolic group. Assuming that $\Gamma_A(G)$ is δ -hyperbolic, let $\gamma > 0$ be such that $\gamma\delta \leq \ln 2$. Following Holopainen, Lang and Vähäkangas [7], we define

$$\rho_{p,\gamma}^A(g, h) = \begin{cases} e^{-\gamma(g|h)_p^A} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $p, g, h \in G$. In general, $\rho_{p,\gamma}^A$ fails to be a metric because of the triangular inequality. Let

$$\sigma_{p,\gamma}^A(g, h) = \inf\{\rho_{p,\gamma}^A(x_0, x_1) + \dots + \rho_{p,\gamma}^A(x_{n-1}, x_n) \mid n \geq 0, x_0 = g, x_n = h; x_1, \dots, x_{n-1} \in G\}.$$

By [7] (cf. also [4, 17]), $\sigma_{p,\gamma}^A$ is a metric on G and the inequalities

$$\frac{1}{4}\rho_{p,\gamma}^A(g, h) \leq \sigma_{p,\gamma}^A(g, h) \leq \rho_{p,\gamma}^A(g, h) \tag{6}$$

hold for all $g, h \in G$.

The metric $\sigma_{p,\gamma}^A$ is an important example of a *visual metric*. Given $p \in G$, $\gamma > 0$ and $T \geq 1$, we denote by $V^A(p, \gamma, T)$ the set of all metrics d on G such that

$$\frac{1}{T}\rho_{p,\gamma}^A(g, h) \leq d(g, h) \leq T\rho_{p,\gamma}^A(g, h) \quad (7)$$

holds for all distinct $g, h \in G$. By (6), we have

$$\sigma_{p,\gamma}^A \in V^A(p, \gamma, 4)$$

whenever $\gamma\delta \leq \ln 2$. We shall refer to the metrics in some $V^A(p, \gamma, T)$ as the *visual metrics* on G .

Let $d \in V^A(p, \gamma, T)$ be a visual metric. In general, the metric space (G, d) is not complete. But its completion is essentially unique and also compact, and can be obtained by adding to G the elements of the boundary ∂G [2, 5, 7, 17]. We denote it by $(\widehat{G}, \widehat{d})$. It is well known that \widehat{d} induces the *Gromov topology* on ∂G [2, Section III.H.3].

To understand the metric \widehat{d} , we must consider the extension of $\rho_{p,\gamma}^A$ to the boundary. We define

$$\hat{\rho}_{p,\gamma}^A(\alpha, \beta) = \begin{cases} e^{-\gamma(\alpha|\beta)_p^A} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in \widehat{G}$. By continuity, the inequalities

$$\frac{1}{T}\hat{\rho}_{p,\gamma}^A(\alpha, \beta) \leq \widehat{d}(\alpha, \beta) \leq T\hat{\rho}_{p,\gamma}^A(\alpha, \beta) \quad (8)$$

hold for all $\alpha, \beta \in \widehat{G}$ [2, Section III.H.3].

It is widely known that uniform continuity of a mapping $\varphi : G \rightarrow G'$ of hyperbolic groups determines the existence of a continuous extension $\Phi : \widehat{G} \rightarrow \widehat{G}'$:

Lemma 3.1 *Let $\varphi : G \rightarrow G'$ be a mapping of hyperbolic groups and let d and d' be visual metrics on G and G' respectively. Then the following conditions are equivalent:*

- (i) φ is uniformly continuous with respect to d and d' ;
- (ii) φ admits a continuous extension $\Phi : (\widehat{G}, \widehat{d}) \rightarrow (\widehat{G}', \widehat{d}')$.

Indeed, by a general topology result [3, Section XIV.6], every uniformly continuous mapping $\varphi : G \rightarrow G'$ admits a continuous extension to the completions.

On the other hand, the completion $(\widehat{G}, \widehat{d})$ is compact. Since every continuous mapping with compact domain is uniformly continuous, it follows that φ , being a restriction of a uniformly continuous extension, is itself uniformly continuous.

We note also that the continuous extension is uniquely defined through

$$[(g_n)_n]\Phi = [(g_n\varphi)_n],$$

for every Gromov sequence $(g_n)_n$ on G .

A group is *virtually free* if it has a free subgroup of finite index. Finitely generated virtually free groups constitute an important subclass of hyperbolic groups. We should mention that the second author developed in [16] a model for the boundary of such a group which allows a huge simplification with respect to the general case.

Lemma 3.2 *Let G be a hyperbolic group and let $d \in V^A(p, \gamma, T)$, $d' \in V^{A'}(p', \gamma', T')$ be visual metrics on G . Let $\varphi : (G, d) \rightarrow (G, d')$ be a mapping and let $P > 0$ and $Q \in \mathbb{R}$ be constants such that*

$$P(g\varphi|h\varphi)_{p'}^{A'} + Q \geq (g|h)_p^A \quad (9)$$

holds for all $g, h \in G$. Then φ satisfies a Hölder condition of exponent $\frac{\gamma'}{\gamma P}$.

Proof. Let $g, h \in G$. We may assume that $g\varphi \neq h\varphi$, hence

$$\begin{aligned} d'(g\varphi, h\varphi) &\leq T' \rho_{p', \gamma'}^{A'}(g\varphi, h\varphi) = T' e^{-\gamma'(g\varphi|h\varphi)_{p'}^{A'}} \leq T' e^{-\frac{\gamma'}{P}((g|h)_p^A - Q)} \\ &= T' e^{\frac{\gamma'Q}{P}} e^{-\frac{\gamma'}{P}(g|h)_p^A} = T' e^{\frac{\gamma'Q}{P}} (e^{-\gamma(g|h)_p^A})^{\frac{\gamma'}{P}} = T' e^{\frac{\gamma'Q}{P}} (\rho_{p, \gamma}^A(g, h))^{\frac{\gamma'}{P}} \\ &\leq T' e^{\frac{\gamma'Q}{P}} (Td(g, h))^{\frac{\gamma'}{P}} = T' e^{\frac{\gamma'Q}{P}} T^{\frac{\gamma'}{P}} (d(g, h))^{\frac{\gamma'}{P}} \end{aligned}$$

and we are done. \square

Replacing (7) by (8), we may use the same argument to prove the following:

Lemma 3.3 *Let G be a hyperbolic group and let $d \in V^A(p, \gamma, T)$, $d' \in V^{A'}(p', \gamma', T')$ be visual metrics on G . Let $\Phi : (\widehat{G}, \widehat{d}) \rightarrow (\widehat{G}, \widehat{d}')$ be a mapping and let $P > 0$ and $Q \in \mathbb{R}$ be constants such that*

$$P(\alpha\Phi|\beta\Phi)_{p'}^{A'} + Q \geq (\alpha|\beta)_p^A$$

holds for all $\alpha, \beta \in \widehat{G}$. Then Φ satisfies a Hölder condition of exponent $\frac{\gamma'}{\gamma P}$.

An endomorphism φ of a group G is *trivial* if $G\varphi = 1$. We show next that in the case of non-trivial endomorphisms, Hölder conditions with respect to visual metrics are equivalent to Lipschitz conditions involving the Gromov product.

Proposition 3.4 *Let G be a hyperbolic group and let $d = \sigma_{p, \gamma}^A$, $d' = \sigma_{p', \gamma'}^{A'}$ be visual metrics on G . Let $\varphi : (G, d) \rightarrow (G, d')$ be a nontrivial homomorphism and let $r > 0$. Then the following conditions are equivalent:*

- (i) φ satisfies a Hölder condition of exponent r ;
- (ii) there exists a constant $Q \in \mathbb{R}$ such that

$$\frac{\gamma'}{r\gamma}(g\varphi|h\varphi)_{p'}^{A'} + Q \geq (g|h)_p^A \quad (10)$$

holds for all $g, h \in G$.

Proof. (i) \Rightarrow (ii). There exists a constant $K > 0$ such that

$$d'(g\varphi, h\varphi) \leq K(d(g, h))^r$$

for all $g, h \in G$.

Assume first that $g\varphi \neq h\varphi$. Then $g \neq h$ and

$$\begin{aligned} e^{-\gamma'(g\varphi|h\varphi)_{p'}^{A'}} &= \rho_{p', \gamma'}^{A'}(g\varphi, h\varphi) \leq T' d'(g\varphi, h\varphi) \leq T' K(d(g, h))^r \leq T' K(T\rho_{p, \gamma}^A(g, h))^r \\ &\leq T' K T^r (\rho_{p, \gamma}^A(g, h))^r = T' K T^r e^{-r\gamma(g|h)_p^A}, \end{aligned}$$

hence

$$-\gamma'(g\varphi|h\varphi)_{p'}^{A'} \leq \ln(T'KT^r) - r\gamma(g|h)_p^A$$

and so

$$\frac{\gamma'}{r\gamma}(g\varphi|h\varphi)_{p'}^{A'} + \frac{\ln(T'KT^r)}{r\gamma} \geq (g|h)_p^A \quad (11)$$

holds whenever $g\varphi \neq h\varphi$.

Now, since φ is nontrivial, there exists some $a \in A$ such that $a\varphi \neq 1$. We may assume that $d_A(1, a\varphi)$ is minimal. We show that (10) holds for

$$Q = 1 + \frac{\ln(T'KT^r)}{r\gamma} + \frac{\gamma'}{r\gamma}d_{A'}(1, a\varphi).$$

In view of (11), we may assume that $g\varphi = h\varphi$. On the one hand, using (11), we have

$$\begin{aligned} (g\varphi|h\varphi)_{p'}^{A'} &= \frac{1}{2}(d_{A'}(p', g\varphi) + d_{A'}(p', h\varphi) - d_{A'}(g\varphi, h\varphi)) \\ &\geq \frac{1}{2}(d_{A'}(p', g\varphi) + d_{A'}(p', (ha)\varphi) - d_{A'}(g\varphi, (ha)\varphi) - 2d_{A'}(h\varphi, (ha)\varphi)) \\ &= (g\varphi|(ha)\varphi)_{p'}^{A'} - d_{A'}(1, a\varphi) \\ &\geq \frac{r\gamma}{\gamma'}(g|ha)_p^A - \frac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1, a\varphi). \end{aligned}$$

On the other hand, $a\varphi \neq 1$ implies $a \neq 1$ and so

$$\begin{aligned} (g|ha)_p^A &= \frac{1}{2}(d_A(p, g) + d_A(p, ha) - d_A(g, ha)) \\ &\geq \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h) - 2d_A(h, ha)) \\ &= (g|h)_p^A - 1, \end{aligned}$$

hence

$$\begin{aligned} (g\varphi|h\varphi)_{p'}^{A'} &\geq \frac{r\gamma}{\gamma'}(g|ha)_p^A - \frac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1, a\varphi) \geq \frac{r\gamma}{\gamma'}((g|h)_p^A - 1) - \frac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1, a\varphi) \\ &= \frac{r\gamma}{\gamma'}((g|h)_p^A - Q) \end{aligned}$$

and so (10) holds as required.

(ii) \Rightarrow (i). By Lemma 3.2. \square

The next technical lemma illustrates an easy way of producing quasi-geodesics. If (X, d) is a geodesic metric space and

$$x_0 \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_n \quad (12)$$

is a path in X such that each $x_{i-1} \longrightarrow x_i$ is a geodesic, then (12) induces a canonical mapping $\xi : [0, s] \rightarrow X$ such that $s = d(x_0, x_1) + \dots + d(x_{n-1}, x_n)$, $0\xi = x_0$ and $s\xi = x_n$.

Lemma 3.5 *Let (X, d) be a geodesic metric space and let $\xi : [0, s] \rightarrow X$ be the canonical mapping induced by*

$$x_0 \longrightarrow x_1 \longrightarrow \dots \longrightarrow x_n,$$

where each $x_{i-1} \longrightarrow x_i$ is a geodesic. Let $P, L > 0$ and $Q \geq 0$ be such that

$$1 \leq d(x_{i-1}, x_i) \leq L \quad (13)$$

and

$$Pd(x_i, x_j) + Q \geq |i - j| \quad (14)$$

for all $i, j \in \{1, \dots, n\}$. Then ξ is a (λ, K) -quasigeodesic for

$$\lambda = \max\{1, LP\}, \quad K = \max\{2L, \frac{Q+1}{P} + L\}.$$

Proof. For $k = 0, \dots, n$, let

$$s_k = \sum_{i=1}^k d(x_{i-1}, x_i).$$

Clearly, $s_k \xi = x_k$. It suffices to show that

$$\frac{|u-v|}{LP} - \frac{Q+1}{P} - L \leq d(u\xi, v\xi) \leq |u-v| + 2L \quad (15)$$

for all $u, v \in [0, s]$.

Indeed, it follows from (13) that there exist some $i, j \in \{0, \dots, n\}$ such that

$$d(u\xi, s_i\xi) = |u - s_i| \leq \frac{L}{2}, \quad d(v\xi, s_j\xi) = |v - s_j| \leq \frac{L}{2}.$$

By symmetry, we may assume that $i \leq j$. Hence

$$\begin{aligned} d(u\xi, v\xi) &\leq d(s_i\xi, s_j\xi) + L = d(x_i, x_j) + L \\ &\leq \sum_{\ell=i+1}^j d(x_{\ell-1}, x_\ell) + L = \sum_{\ell=i+1}^j (s_\ell - s_{\ell-1}) + L \\ &= s_j - s_i + L \leq |u - v| + 2L. \end{aligned}$$

On the other hand, (13) and (14) yield

$$\begin{aligned} d(u\xi, v\xi) &\geq d(s_i\xi, s_j\xi) - L = d(x_i, x_j) - L \\ &\geq \frac{1}{P}|i-j| - \frac{Q}{P} - L \geq \frac{1}{P} \sum_{\ell=i+1}^j \frac{s_\ell - s_{\ell-1}}{L} - \frac{Q}{P} - L \\ &= \frac{s_j - s_i}{LP} - \frac{Q}{P} - L \geq \frac{|u-v|}{LP} - \frac{Q+1}{P} - L \end{aligned}$$

and so (15) holds as required. \square

Two metrics d and d' on a set X are *Hölder equivalent* if the identity mappings $(X, d) \rightarrow (X, d')$ and $(X, d') \rightarrow (X, d)$ satisfy both a Hölder condition.

The following proposition is the finite version of the well-known analogue result on the equivalence of the visual metrics on the boundary (see [9, Theorem 2.18]).

Proposition 3.6 *All visual metrics on a given hyperbolic group are Hölder equivalent.*

Proof. Let G be a hyperbolic group and let d, d' be visual metrics on G . Let A, A' be finite generating sets of G and assume that $\bar{\Gamma}_A(G)$ (respectively $\bar{\Gamma}_{A'}(G)$) is δ -hyperbolic (respectively δ' -hyperbolic). Let $p, p' \in G$. In view of Proposition 3.4, it suffices to show that there exist constants $P > 0$ and $Q \geq 0$ such that

$$P(g|h)_{p'}^{A'} + Q \geq (g|h)_p^A \quad (16)$$

holds for all $g, h \in G$.

Let $N = N_{A, A'}$ be as in (3) and let $R = R(\delta, N^2, 2N)$ be the constant introduced in Section 2. Let $[g, h]_A$ and $[g, h]_{A'}$ be geodesics in $\bar{\Gamma}_A(G)$ and $\bar{\Gamma}_{A'}(G)$, respectively. We claim that

$$d_A(p, [g, h]_A) \leq Nd_{A'}(p', [g, h]_{A'}) + N + d_A(p, p') + R. \quad (17)$$

Assume that $[g, h]_{A'}$ is the path

$$g = g_0 \xrightarrow{a'_1} g_1 \xrightarrow{a'_2} \dots \xrightarrow{a'_n} g_n = h$$

with $a'_1, \dots, a'_n \in \widetilde{A'}$. Consider geodesics $g_{i-1} \xrightarrow{u_i} g_i$ in $\overline{\Gamma}_A(G)$ and let $\xi : [0, s] \rightarrow (\overline{\Gamma}_A(G), d_A)$ be the canonical mapping induced by the path

$$g = g_0 \xrightarrow{u_1} g_1 \xrightarrow{u_2} \dots \xrightarrow{u_n} g_n = h.$$

Then $1 \leq d_A(g_{i-1}, g_i) \leq N$ and in view of (4)

$$Nd_A(g_i, g_j) \geq d_{A'}(g_i, g_j) = |i - j|$$

holds for all $i, j \in \{1, \dots, n\}$. By Lemma 3.5, ξ is a $(N^2, 2N)$ -quasi-geodesic. Note that $0\xi = g$, $s\xi = h$ and $[g, h]_{A'} \cap G \subseteq \text{Im}(\xi)$.

Now

$$\text{Haus}([g, h]_A, \text{Im}(\xi)) \leq R(\delta, N^2, 2N) = R,$$

hence

$$d_A(p, [g, h]_A) \leq d_A(p, \text{Im}(\xi)) + R \leq d_A(p, p') + d_A(p', \text{Im}(\xi)) + R. \quad (18)$$

On the other hand, we have

$$d_{A'}(p', [g, h]_{A'}) \geq d_{A'}(p', [g, h]_{A'} \cap G) - 1$$

and

$$d_A(p', \text{Im}(\xi)) \leq d_A(p', \text{Im}(\xi) \cap G) \leq d_A(p', [g, h]_{A'} \cap G)$$

follows from $[g, h]_{A'} \cap G \subseteq \text{Im}(\xi)$. In view of (4), we get

$$d_{A'}(p', [g, h]_{A'}) \geq d_{A'}(p', [g, h]_{A'} \cap G) - 1 \geq \frac{1}{N}d_A(p', [g, h]_{A'} \cap G) - 1 \geq \frac{1}{N}d_A(p', \text{Im}(\xi)) - 1.$$

Together with (18), this yields (17).

By [17, Lemmas 2.9, 2.31 and 2.32], we have

$$(g|h)_p^A \leq d_A(p, [g, h]_A) \leq (g|h)_p^A + 2\delta. \quad (19)$$

Together with (17), this yields

$$(g|h)_p^A \leq Nd_{A'}(p', [g, h]_{A'}) + N + d_A(p, p') + R.$$

Applying (19) to $d_{A'}(p', [g, h]_{A'})$, we obtain

$$(g|h)_p^A \leq N(g|h)_{p'}^{A'} + 2N\delta' + N + d_A(p, p') + R,$$

hence (16) holds for $P = N$ and $Q = (2\delta' + 1)N + d_A(p, p') + R$. \square

4 Endomorphisms of hyperbolic groups

An endomorphism φ of G is *virtually injective* if its kernel is finite. This is a necessary condition for uniform continuity:

Lemma 4.1 *Let G be a hyperbolic group endowed with a visual metric d . Let φ be a uniformly continuous nontrivial endomorphism of G . Then φ is virtually injective.*

Proof. Assume that $d \in V^A(p, \gamma, T)$. Fix $g \in G \setminus \text{Ker}(\varphi)$. Let $\varepsilon = d(1, g\varphi) > 0$ and let $\delta > 0$ be such that

$$\forall x, y \in G \ (d(x, y) < \delta \Rightarrow d(x\varphi, y\varphi) < \varepsilon).$$

For every $h \in \text{Ker}(\varphi)$, we have $d(h\varphi, (hg)\varphi) = d(1, g\varphi) = \varepsilon$, hence $d(h, hg) \geq \delta$. By (7), we get

$$e^{-\gamma(h|hg)_p} = \rho_{p,\gamma}^A(h, hg) \geq \frac{1}{T}d(h, hg) \geq \frac{\delta}{T}$$

and so

$$(h|hg)_p \leq -\frac{\ln \frac{\delta}{T}}{\gamma}.$$

It follows that

$$\begin{aligned} d_A(p, h) &\leq \frac{1}{2}(d_A(p, h) + d_A(p, hg) - d_A(h, hg) + 2d_A(h, hg)) = (h|hg)_p + d_A(h, hg) \\ &= (h|hg)_p + d_A(1, g) \leq -\frac{\ln \frac{\delta}{T}}{\gamma} + d_A(1, g). \end{aligned}$$

Since A is finite, then $\Gamma_A(G)$ is locally finite, i.e. every ball is finite. Therefore $\text{Ker}(\varphi)$ is finite and φ is virtually injective. \square

We need also the following result:

Proposition 4.2 *Let φ be a nontrivial endomorphism of a hyperbolic group G with continuous extension $\Phi : \widehat{G} \rightarrow \widehat{G}$. Let d be a visual metric on G . Then the following conditions are equivalent:*

- (i) φ satisfies a Hölder condition of exponent r with respect to d ;
- (ii) Φ satisfies a Hölder condition of exponent r with respect to \widehat{d} .

Proof. (i) \Rightarrow (ii). Let $d \in V^A(p, \gamma, T)$. By Proposition 3.4, there exists a constant $Q \in \mathbb{R}$ such that

$$\frac{1}{r}(g\varphi|h\varphi)_p^A + Q \geq (g|h)_p^A \quad (20)$$

holds for all $g, h \in G$. We show that

$$\frac{1}{r}(\alpha\Phi|\beta\Phi)_p^A + Q \geq (\alpha|\beta)_p^A \quad (21)$$

for all $\alpha, \beta \in \widehat{G}$.

Let $(g_n)_n \in \alpha$ and $(h_n)_n \in \beta$. For every $n \in \mathbb{N}$, we have by (20)

$$\inf\{(g_i\varphi|h_j\varphi)_p^A \mid i, j \geq n\} \geq r \cdot \inf\{(g_i|h_j)_p^A \mid i, j \geq n\} - rQ.$$

Hence

$$\lim_{i,j \rightarrow +\infty} (g_i \varphi | h_j \varphi)_p^A \geq r \lim_{i,j \rightarrow +\infty} (g_i | h_j)_p^A - rQ.$$

It follows that

$$\sup \{ \lim_{i,j \rightarrow +\infty} (g_i \varphi | h_j \varphi)_p^A \mid (g_n)_n \in \alpha, (h_n)_n \in \beta \} \geq r(\alpha | \beta)_p^A - rQ.$$

Since $(g_n)_n \in \alpha$ implies $(g_n \varphi)_n \in \alpha \Phi$ and $(h_n)_n \in \beta$ implies $(h_n \varphi)_n \in \beta \Phi$, it follows that

$$(\alpha \Phi | \beta \Phi)_p^A \geq r(\alpha | \beta)_p^A - rQ.$$

By Lemma 3.3, Φ satisfies a Hölder condition of exponent r with respect to \widehat{d} .

(ii) \Rightarrow (i). Immediate since Φ is an extension of φ and \widehat{d} is an extension of d . \square

In the main result of the paper, we characterize the uniformly continuous endomorphisms which satisfy a Hölder condition:

Theorem 4.3 *Let φ be a nontrivial endomorphism of a hyperbolic group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G . Then the following conditions are equivalent:*

- (i) φ satisfies a Hölder condition with respect to d ;
- (ii) φ admits an extension to \widehat{G} satisfying a Hölder condition with respect to \widehat{d} ;
- (iii) there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that

$$P(g\varphi | h\varphi)_p^A + Q \geq (g | h)_p^A \quad (22)$$

for all $g, h \in G$;

- (iv) φ is a quasi-isometric embedding of (G, d_A) into itself;
- (v) φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G .

Proof. (i) \Leftrightarrow (ii). By Lemma 3.1 and Proposition 4.2.

(i) \Leftrightarrow (iii). By Proposition 3.4.

(i) \Rightarrow (v). By Lemma 4.1, φ is virtually injective. In view of Proposition 3.6, we may assume that $p = 1$. Since (i) implies (iii), there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that

$$P(g\varphi | g\varphi)_1^A + Q \geq (g | g)_1^A$$

for every $g \in G$, which is equivalent to

$$Pd_A(1, g\varphi) + Q \geq d_A(1, g).$$

Since $d_A(g, h) = d_A(1, g^{-1}h)$ and $d_A(g\varphi, h\varphi) = d_A(1, (g^{-1}h)\varphi)$, we immediately get

$$Pd_A(g\varphi, h\varphi) + Q \geq d_A(g, h) \quad (23)$$

for all $g, h \in G$.

Let

$$M_\varphi = \max\{d_A(1, a\varphi) \mid a \in A\}.$$

We show now that $G\varphi$ is quasiconvex.

Let $g, h \in G$ and let $g' \xrightarrow{w} h'$ have minimal length among all the paths in $\bar{\Gamma}_A(G)$ such that $g'\varphi = g\varphi$ and $h'\varphi = h\varphi$. In particular, $g' \xrightarrow{w} h'$ is a geodesic. Assume that $w = a_1 \dots a_n$ with $a_i \in \tilde{A}$. For $i = 0, \dots, n$, write $w_i = a_1 \dots a_i$ and let $(g'w_{i-1})\varphi \rightarrow (g'w_i)\varphi$ be a geodesic. Let $\xi : [0, s] \rightarrow \bar{\Gamma}_A(G)$ be the canonical mapping induced by the path

$$g\varphi = g'\varphi = (g'w_0)\varphi \rightarrow (g'w_1)\varphi \rightarrow \dots \rightarrow (g'w_n)\varphi = h'\varphi = h\varphi.$$

Suppose that $a_i\varphi = 1$ for some i . Let $w' = w_{i-1}a_{i+1} \dots a_n$. Since $w'\varphi = w\varphi$, we have $(g'w')\varphi = (g'w)\varphi = h'\varphi = h\varphi$, contradicting the minimality of w . Hence $a_i\varphi \neq 1$ for every i . Since $|a_i\varphi| \leq M_\varphi$, we get

$$1 \leq d_A((g'w_{i-1})\varphi, (g'w_i)\varphi) \leq M_\varphi.$$

Assume that $0 \leq i \leq j \leq n$. By (23), we have

$$Pd_A((g'w_i)\varphi, (g'w_j)\varphi) + Q \geq d_A(g'w_i, g'w_j) = d_A(1, a_{i+1} \dots a_j).$$

Since $a_{i+1} \dots a_j$ is a factor of a geodesic, it is itself a geodesic and so $d_A(1, a_{i+1} \dots a_j) = j - i$. Thus

$$Pd_A((g'w_i)\varphi, (g'w_j)\varphi) + Q \geq |j - i|$$

and it follows from Lemma 3.5 that ξ is a (λ, K) -quasigeodesic for

$$\lambda = \max\{1, M_\varphi P\}, \quad K = \max\{2M_\varphi, \frac{Q+1}{P} + M_\varphi\}.$$

Let $R = R(\delta, \lambda, K)$. Let $[g\varphi, h\varphi]$ be a geodesic in $\bar{\Gamma}_A(G)$ and let $x \in [g\varphi, h\varphi]$. Then

$$\text{Haus}([g\varphi, h\varphi], \text{Im}(\xi)) \leq R$$

and every point in $\text{Im}(\xi)$ is at distance at most M_φ from an element of $G\varphi$, hence

$$d_A(x, G\varphi) \leq R + M_\varphi$$

and so $G\varphi$ is $(R + M_\varphi)$ -quasiconvex.

(v) \Rightarrow (iv). Let $K = \text{Ker}(\varphi) \trianglelefteq G$. Let $\pi : G \rightarrow G/K$ be the canonical projection and let $\iota : G\varphi \rightarrow G$ be inclusion. Then there exists an isomorphism $\bar{\varphi} : G/K \rightarrow G\varphi$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \pi \downarrow & & \uparrow \iota \\ G/K & \xrightarrow{\bar{\varphi}} & G\varphi \end{array}$$

commutes. Since the composition of quasi-isometric embeddings is still a quasi-isometric embedding, it suffices to show that each one of the homomorphisms $\pi, \bar{\varphi}, \iota$ is a quasi-isometric embedding when we consider a geodesic metric in each of the groups (it does not matter which since the identity $(H, d_A) \rightarrow (H, d_B)$ is a quasi-isometry whenever $H = \langle A \rangle = \langle B \rangle$ by (4)).

Let

$$L = \max\{d_A(1, x) \mid x \in K\}.$$

Let $g, h \in G$. We claim that

$$d_A(g, h) - L \leq d_{A\pi}(g\pi, h\pi) \leq d_A(g, h). \quad (24)$$

Since $h = ga_1 \dots a_n$ implies $h\pi = (ga_1 \dots a_n)\pi$ for all $a_1, \dots, a_n \in \tilde{A}$, we have $d_{A\pi}(g\pi, h\pi) \leq d_A(g, h)$.

Write $h\pi = (gw)\pi$, where w is a word on \tilde{A} of minimum length. Then $h = gwx$ for some $x \in K$ and so

$$d_A(g, h) \leq d_A(g, gw) + d_A(gw, gwx) = d_A(1, w) + d_A(1, x) \leq |w| + L.$$

By minimality of w , we have actually $|w| = d_{A\pi}(g\pi, h\pi)$ and thus (24) holds.

Now $\bar{\varphi}$ is an isomorphism and $A\pi\bar{\varphi} = A\varphi$, hence

$$d_{A\varphi}(g\pi\bar{\varphi}, h\pi\bar{\varphi}) = d_{A\pi}(g\pi, h\pi)$$

for all $g, h \in G$ and so $\bar{\varphi} : (G/K, d_{A\pi}) \rightarrow (G\varphi, d_{A\varphi})$ is actually an isometry.

Assume that $G\varphi$ is q -quasi convex with respect to A . Let

$$B = \{h \in G\varphi \mid d_A(1, h) \leq 2q + 1\}.$$

Then B is a finite generating set of $G\varphi$ and $\iota : (G\varphi, d_B) \rightarrow (G, d_A)$ is a quasi-isometric embedding [2, Section III.Γ.3]. Therefore all three homomorphisms $\pi, \bar{\varphi}, \iota$ are quasi-isometric embeddings and so is their composition φ .

(iv) \Rightarrow (iii). Clearly, φ can be extended to a quasi-isometric embedding $\bar{\varphi}$ of $(\bar{\Gamma}_A(G), d_A)$ into itself. Assume that $\bar{\Gamma}_A(G)$ is δ -hyperbolic. Let $\lambda \geq 1$ and $K \geq 0$ be constants such that

$$\frac{1}{\lambda}d_A(x, y) - K \leq d_A(x\bar{\varphi}, y\bar{\varphi}) \leq \lambda d_A(x, y) + K \quad (25)$$

holds for all $x, y \in \bar{\Gamma}_A(G)$. Write $R = R(\delta, \lambda, K)$. We prove that

$$\lambda(g\varphi|h\varphi)_p^A + \delta + \lambda(\lambda\delta + \frac{3K}{2} + 3R + d_A(p, p\varphi)) \geq (g|h)_p^A \quad (26)$$

holds for all $g, h \in G$.

Let $g, h \in G$. Consider a geodesic triangle $[[p, g, h]]$ with geodesics $[p, g]$, $[g, h]$ and $[p, h]$. Let

$$X = \{x \in [g, h] : d_A(x, [p, g]) \leq \delta\}, \quad Y = \{y \in [g, h] : d_A(y, [p, h]) \leq \delta\}.$$

It is immediate that X and Y are both closed and nonempty. Since $X \cup Y = [g, h]$ is obviously connected, it follows that $X \cap Y \neq \emptyset$. Let $x \in X \cap Y$ and take $g' \in [p, g]$ and $h' \in [p, h]$ such that $d_A(x, g') \leq \delta$ and $d_A(x, h') \leq \delta$.

If $\xi : [0, s] \rightarrow [p, g]$ is our geodesic, let $\xi' = \xi\bar{\varphi}$. For all $i, j \in [0, s]$, (25) yields

$$d_A(i\xi' - j\xi') = d_A(i\xi\bar{\varphi} - j\xi\bar{\varphi}) \leq \lambda|i\xi - j\xi| + K = \lambda|i - j| + K.$$

Similarly,

$$d_A(i\xi' - j\xi') \geq \frac{1}{\lambda}|i - j| - K$$

and so ξ' is a (λ, K) -quasi-geodesic from $[0, s]$ to $\bar{\Gamma}_A(G)$ such that $0\xi' = p\varphi$, $s\xi' = g\varphi$ and $g'\varphi \in \text{Im}(\xi')$. Fix a geodesic $[p\varphi, g\varphi]$. Then $\text{Haus}([p\varphi, g\varphi], \text{Im}(\xi')) \leq R$, hence there exists some $g'' \in [p\varphi, g\varphi]$ such that $d_A(g'', g'\varphi) \leq R$. Similarly, fix geodesics $[p\varphi, h\varphi]$ and $[g\varphi, h\varphi]$. Then there exist some $h'' \in [p\varphi, h\varphi]$ and $x' \in [g\varphi, h\varphi]$ such that $d_A(h'', h'\varphi), d_A(x', x\varphi) \leq R$.

We claim that

$$d_A(g'', g\varphi) - d_A(g\varphi, x') \geq -\lambda\delta - K - 2R. \quad (27)$$

Indeed, in view of (25), we have

$$\begin{aligned} d_A(g'', g\varphi) - d_A(g\varphi, x') &\geq -d_A(g'', x') \geq -d_A(g'', g'\varphi) - d_A(g'\varphi, x\varphi) - d_A(x\varphi, x') \\ &\geq -\lambda d_A(g', x) - K - 2R \geq -\lambda\delta - K - 2R. \end{aligned}$$

Similarly,

$$d_A(h'', h\varphi) - d_A(h\varphi, x') \geq -\lambda\delta - K - 2R. \quad (28)$$

Now (25), (27) and (28) combined yield

$$\begin{aligned} (g\varphi|h\varphi)_{p\varphi}^A &= \frac{1}{2}(d_A(p\varphi, g\varphi) + d_A(p\varphi, h\varphi) - d_A(g\varphi, h\varphi)) \\ &= \frac{1}{2}(d_A(p\varphi, g'') + d_A(g'', g\varphi) + d_A(p\varphi, h'') + d_A(h'', h\varphi) - d_A(g\varphi, x') - d_A(x', h\varphi)) \\ &= \frac{1}{2}(d_A(p\varphi, g'') + d_A(p\varphi, h'') + d_A(g'', g\varphi) - d_A(g\varphi, x') + d_A(h'', h\varphi) - d_A(x', h\varphi)) \\ &\geq \frac{1}{2}(d_A(p\varphi, g'\varphi) + d_A(p\varphi, h'\varphi) + d_A(g'', g\varphi) - d_A(g\varphi, x') + d_A(h'', h\varphi) - d_A(x', h\varphi)) - R \\ &\geq \frac{1}{2}(d_A(p\varphi, g'\varphi) + d_A(p\varphi, h'\varphi)) - \lambda\delta - K - 3R \\ &\geq \frac{1}{2\lambda}(d_A(p, g') + d_A(p, h')) - \lambda\delta - \frac{3K}{2} - 3R. \end{aligned}$$

It follows that

$$\begin{aligned} (g\varphi|h\varphi)_p^A &= \frac{1}{2}(d_A(p, g\varphi) + d_A(p, h\varphi) - d_A(g\varphi, h\varphi)) \\ &\geq \frac{1}{2}(d_A(p\varphi, g\varphi) + d_A(p\varphi, h\varphi) - 2d_A(p, p\varphi) - d_A(g\varphi, h\varphi)) \\ &= (g\varphi|h\varphi)_{p\varphi}^A - d_A(p, p\varphi) \\ &\geq \frac{1}{2\lambda}(d_A(p, g') + d_A(p, h')) - \lambda\delta - \frac{3K}{2} - 3R - d_A(p, p\varphi). \end{aligned}$$

On the other hand,

$$\begin{aligned} (g|h)_p^A &= \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)) \\ &= \frac{1}{2}(d_A(p, g') + d_A(g', g) + d_A(p, h') + d_A(h', h) - d_A(g, x) - d_A(x, h)) \\ &= \frac{1}{2}(d_A(p, g') + d_A(p, h') + d_A(g', g) - d_A(g, x) + d_A(h', h) - d_A(x, h)) \\ &\leq \frac{1}{2}(d_A(p, g') + d_A(p, h') + d_A(g', x) + d_A(h', x)) \\ &\leq \frac{1}{2}(d_A(p, g') + d_A(p, h')) + \delta \end{aligned}$$

and combining the previous inequalities we get

$$\begin{aligned} (g\varphi|h\varphi)_p^A &\geq \frac{1}{2\lambda}(d_A(p, g'\varphi) + d_A(p, h'\varphi)) - \lambda\delta - \frac{3K}{2} - 3R - d_A(p, p\varphi) \\ &\geq \frac{1}{\lambda}(g|h)_p^A - \frac{\delta}{\lambda} - \lambda\delta - \frac{3K}{2} - 3R - d_A(p, p\varphi). \end{aligned}$$

Therefore (26) holds and we are done. \square

5 Simplifications

Under which circumstances does every uniformly continuous endomorphism satisfy a Hölder condition? We have the following remark:

Lemma 5.1 *Let G be a hyperbolic group and let $d \in V^A(p, \gamma, T)$ be a visual metric on G . Then the following conditions are equivalent:*

- (i) *every uniformly continuous endomorphism of G satisfies a Hölder condition with respect to d ;*
- (ii) *$G\varphi$ is a quasiconvex subgroup of G for every endomorphism of G uniformly continuous with respect to d .*

Proof. (i) \Rightarrow (ii). Both conditions hold for the trivial endomorphism. For nontrivial endomorphisms we use Theorem 4.3.

(ii) \Rightarrow (i). Let φ be a nontrivial endomorphism of G , uniformly continuous with respect to d . By Lemma 4.1, φ is virtually injective. Now we apply Theorem 4.3. \square

Now we get a simplified version of Theorem 4.3 for virtually free groups:

Corollary 5.2 *Let φ be a nontrivial endomorphism of a finitely generated virtually free group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G . Then the following conditions are equivalent:*

- (i) *φ is uniformly continuous with respect to d ;*
- (ii) *φ satisfies a Hölder condition with respect to d ;*
- (iii) *φ admits a continuous extension to the completion $(\widehat{G}, \widehat{d})$;*
- (iv) *φ admits an extension to \widehat{G} satisfying a Hölder condition with respect to \widehat{d} ;*
- (v) *there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that*

$$P(g\varphi|h\varphi)_p^A + Q \geq (g|h)_p^A$$

for all $g, h \in G$;

- (vi) *φ is a quasi-isometric embedding of (G, d_A) into itself;*
- (vii) *φ is virtually injective.*

Proof. By [1, Corollary 4.2], every subgroup of a finitely generated virtually free group is quasiconvex, hence the equivalences (ii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) follow from Theorem 4.3. Now (ii) \Rightarrow (i) holds trivially, (i) \Rightarrow (vii) follows from Lemma 4.1, and (i) \Leftrightarrow (iii) follows from Lemma 3.1. \square

We note that the equivalence (i) \Leftrightarrow (iii) \Leftrightarrow (vii) had been proved by the second author in [16].

If φ is not an endomorphism, then Corollary 5.2 does not hold, even for an infinite cyclic group. For $x \in \mathbb{R}$, we denote by $[x]$ the greatest integer $n \leq x$.

Example 5.3 *Let d be the prefix metric on $F_{\{a\}}$ and let $\varphi : F_{\{a\}} \rightarrow F_{\{a\}}$ be defined by*

$$a^n \varphi = \begin{cases} a^{\lfloor \sqrt{n} \rfloor} & \text{if } n \geq 0 \\ a^n & \text{otherwise} \end{cases}$$

Then φ is uniformly continuous but satisfies no Hölder condition with respect to d .

Indeed, it is a simple exercise to show that

$$\forall \varepsilon > 0 \forall m, n \in \mathbb{Z} (d(a^m, a^n) < \min\{\frac{1}{2}, \frac{1}{2}\varepsilon^{2-\log_2 \varepsilon}\} \Rightarrow d(a^m \varphi, a^n \varphi) < \varepsilon),$$

hence φ is uniformly continuous with respect to d .

However, for all $r, K > 0$, we have

$$rm - \lfloor \sqrt{m} \rfloor > \log_2 K$$

for m large enough. It is easy to check that, for such m and $n > (\sqrt{m} + 1)^2$, we have

$$d(a^m \varphi, a^n \varphi) > K(d(a^m, a^n))^r,$$

thus φ satisfies no Hölder condition with respect to d .

We would like to extend the previous discussion to the class of all hyperbolic groups, but the panorama is not so clear. But we are able to produce another result on the line of Corollary 5.2. We recall that a group G is said to be *co-hopfian* if every monomorphism of G is an automorphism.

Corollary 5.4 *Let φ be a nontrivial endomorphism of a torsion-free co-hopfian hyperbolic group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G . Then the following conditions are equivalent:*

- (i) φ is uniformly continuous with respect to d ;
- (ii) φ satisfies a Hölder condition with respect to d ;
- (iii) φ admits a continuous extension to the completion $(\widehat{G}, \widehat{d})$;
- (iv) φ admits an extension to \widehat{G} satisfying a Hölder condition with respect to \widehat{d} ;
- (v) there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that

$$P(g\varphi|h\varphi)_p^A + Q \geq (g|h)_p^A$$

for all $g, h \in G$;

- (vi) φ is a quasi-isometric embedding of (G, d_A) into itself;
- (vii) φ is virtually injective;
- (viii) φ is an automorphism.

Proof. (vii) \Leftrightarrow (viii). Since G is torsion-free, every virtually injective endomorphism is a monomorphism. Then we use the fact that G is co-hopfian.

Now the equivalences (ii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) follow from Theorem 4.3.

Finally, (ii) \Rightarrow (i) holds trivially, (i) \Rightarrow (vii) follows from Lemma 4.1, and (i) \Leftrightarrow (iii) follows from Lemma 3.1. \square

Examples of torsion-free co-hopfian hyperbolic groups have been provided by Rips and Sela [13] and Sela [14], namely:

- non-elementary torsion-free hyperbolic groups which admit no nontrivial cyclic splittings [13];
- non-elementary torsion-free freely indecomposable hyperbolic groups [14].

However, many torsion-free hyperbolic groups fail to be co-hopfian, such as infinite cyclic groups. For more interesting examples, see [11].

In the case of torsion-free groups, it would be enough of course to prevent the existence of monomorphisms with non quasiconvex image. The following example shows that such a situation cannot always be avoided.

Example 5.5 *There exists a torsion-free hyperbolic group G having a non quasiconvex subgroup isomorphic to G .*

Indeed, let a, b, t be distinct letters and write $A = \{a, b\}$ and $B = \{a, b, t\}$. We fix words $u = abab^2 \dots ab^{20}$ and $v = baba^2 \dots ba^{20}$. Let H be the group defined by the presentation

$$\langle B \mid t^{-1}atu, t^{-1}btv \rangle. \quad (29)$$

Let R denote the set of all cyclic conjugates of the two (cyclically reduced) relators and their inverses. A *piece* of (29) is a maximal common prefix of two distinct elements of R . It is easy to see that the longest pieces of (29) are $b^{18}ab^{19}$, $a^{18}ab^{19}$ and their inverses and have therefore length 38. On the other hand, the length of each relator is $3 + 20 + \frac{20(20+1)}{2} = 233$. Since $38 < \frac{1}{6}233$, the presentation (29) satisfies the small cancellation condition $C'(\frac{1}{6})$. Now it follows from a theorem of Gromov [6] that H is hyperbolic.

Let F denote the subgroup of H generated by a, b . The subgroup K of F_A generated by u and v cannot have rank 1 since $uv \neq vu$ in F_A . Since F_A is hopfian, it follows that K is free on $\{u, v\}$. Hence H is an HNN extension of F_A and so there is a canonical isomorphism $F_A \rightarrow F$. Thus F is a free subgroup of H with basis A . Moreover, since the finite order elements of the HNN extension H must be conjugates of the finite order elements of F (see [12]), then H is torsion-free.

Since F is a normal subgroup of H , we have $t^{-n}at^n \in F$ for every $n \geq 0$. Consider the geodesic metrics d_A and d_B on F and H . We have

$$d_B(1, t^{-n}at^n) \leq 2n + 1 \quad (30)$$

for every $n \geq 0$. We prove that

$$d_A(1, t^{-n}at^n) = 230^n \quad (31)$$

by induction on n . The case $n = 0$ being trivial, assume that $n \geq 1$ and $d_A(1, t^{-(n-1)}at^{n-1}) = 230^{n-1}$. It follows that there exist $m = 230^{n-1}$ letters $c_1, \dots, c_m \in \tilde{A}$ such that $t^{-(n-1)}at^{n-1} = c_1 \dots c_m$ in reduced form. Hence

$$t^{-n}at^n = (t^{-1}c_1t) \dots (t^{-1}c_mt).$$

We have $t^{-1}c_it \in \{u, v, u^{-1}, v^{-1}\}$ for $i = 1, \dots, m$. Moreover, since $u = a \dots b$ and $v = b \dots a$, and $c_1 \dots c_m$ is reduced, there is no reduction between the reduced forms over \tilde{A} of two consecutive $t^{-1}c_it$. Hence the length of the reduced form of $t^{-n}at^n$ over \tilde{A} is $230m = 230^n$. Since F is free on A , we get (31).

Now it follows from (30) and (31) that the embedding $(F, d_A) \rightarrow (H, d_B)$ is not a quasi-isometric embedding and so F is not an *undistorted* subgroup of H . By a theorem of Short [15] (see also [2, Lemma 7.3.5]), F is a non quasiconvex subgroup of H .

Consider now the free product $G = H * F$. Since it is well known that hyperbolic groups are closed under free product, G is hyperbolic. Moreover, being a free product of torsion-free groups, it is torsion-free as well. Let $K = \langle H \cup aHa^{-1} \rangle \leq G$ (where a comes from the second factor in $H * F$). It is easy to check that

$$K \cong H * H. \quad (32)$$

Indeed, we define a homomorphism $\varphi : H * H \rightarrow K$ by sending an element h from the first factor H into $h \in K$, and an element h from the second factor H into $aha^{-1} \in K$. This is clearly surjective, and injectivity follows from the free product normal form.

We consider now the sequence of embeddings

$$G = H * F \xrightarrow{\theta} H * H \xrightarrow{\varphi} K < H * F = G \quad (33)$$

where θ acts as the identity $H \rightarrow H$ with respect to the first factors and as the inclusion $F \rightarrow H$ for the second ones. Using indices 1 and 2 to distinguish generators from different free factors in the free products, we fix now the finite generating sets C and D for the G and $H * H$, respectively:

$$C = \{a_1, b_1, t_1, a_2, b_2\}, \quad D = \{a_1, b_1, t_1, a_2, b_2, t_2\}.$$

Let d_C and d_D denote the corresponding geodesic metrics on G and $H * H$, respectively.

For each $n \geq 0$, let w_n denote the (unique) reduced word over $\widetilde{\{a_2, b_2\}}$ representing the element $t_2^{-n} a_2 t_2^n \in F$. It follows from (31) and the free product normal form that

$$d_C(w_n) = 230^n. \quad (34)$$

Now by (30) we have $d_D(w_n \theta) \leq 2n + 1$. We claim that

$$d_C(w_n \theta \varphi) \leq 6n + 3. \quad (35)$$

Indeed, each generator a_2, b_2, t_2 of $H * H$ is sent by φ into $a_2 a_1 a_2^{-1}, a_2 b_1 a_2^{-1}, a_2 t_1 a_2^{-1}$, respectively, and so $d_D(w_n \theta) \leq 2n + 1$ yields $d_C(w_n \theta \varphi) \leq 3(2n + 1) = 6n + 3$. Thus (35) holds. Together with (34), this implies that (33) is not a quasi-isometric embedding. By the aforementioned theorem of Short, G has a non quasiconvex subgroup isomorphic to itself.

If the embedding in such an example can be taken to be uniformly continuous with respect to some visual metric, we shall have proved that quasiconvexity cannot be removed from condition (v) in Theorem 4.3(v). But we have no answer yet.

6 Lipschitz conditions

An endomorphism φ of a hyperbolic group $G = \langle A \rangle$ satisfies an obvious Lipschitz condition if G is finite. The following two results provide other instances of Lipschitz conditions.

Given $x \in G$, we denote by λ_x the inner automorphism of G defined by $g\lambda_x = x^{-1}gx$.

Proposition 6.1 *Let φ be an inner automorphism of a hyperbolic group G and let d be a visual metric on G . Then φ satisfies a Lipschitz condition.*

Proof. Write $\varphi = \lambda_x$. Let $d \in V^A(p, \gamma, T)$. In view of Proposition 3.4, it suffices to prove that there exists a constant $Q \in \mathbb{R}$ such that

$$(x^{-1}gx|x^{-1}hx)_p^A + Q \geq (g|h)_p^A \quad (36)$$

holds for all $g, h \in G$.

Now

$$\begin{aligned} (x^{-1}gx|x^{-1}hx)_p^A &= \frac{1}{2}(d_A(p, x^{-1}gx) + d_A(p, x^{-1}hx) - d_A(x^{-1}gx, x^{-1}hx)) \\ &= \frac{1}{2}(d_A(xp, gx) + d_A(xp, hx) - d_A(gx, hx)) \\ &\geq \frac{1}{2}(d_A(p, g) - d_A(p, xp) - d_A(g, gx) + d_A(p, h) - d_A(p, xp) \\ &\quad - d_A(h, hx) - d_A(g, h) - d_A(g, gx) - d_A(h, hx)) \\ &= (g|h)_p^A - 2d(1, x) - d(p, xp) \end{aligned}$$

and so (36) holds for $Q = 2d(1, x) + d(p, xp)$. \square

Lemma 6.2 *Let φ be an automorphism of a hyperbolic group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G . If $\varphi : (G, d_A) \rightarrow (G, d_A)$ is an isometry, then φ satisfies a Lipschitz condition with respect to d .*

Proof. In view of Proposition 3.4, it suffices to prove that there exists a constant $Q \in \mathbb{R}$ such that

$$(g\varphi|h\varphi)_p^A + Q \geq (g|h)_p^A \quad (37)$$

holds for all $g, h \in G$.

Since

$$\begin{aligned} (g\varphi|h\varphi)_p^A &= \frac{1}{2}(d_A(p, g\varphi) + d_A(p, h\varphi) - d_A(g\varphi, h\varphi)) \\ &\geq \frac{1}{2}(d_A(p\varphi, g\varphi) + d_A(p\varphi, h\varphi) - 2d_A(p, p\varphi) - d_A(g\varphi, h\varphi)) \\ &= \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)) - d_A(p, p\varphi) = (g|h)_p^A - d_A(p, p\varphi), \end{aligned}$$

then (37) holds for $Q = d_A(p, p\varphi)$. \square

We consider next the particular case of free groups. Given $g \in F_A$, we denote by $|g|_c$ the *cyclic length* of g , i.e. the length of a cyclically reduced conjugate of g .

Lemma 6.3 *Let $d \in V^A(p, \gamma, T)$ be a visual metric on F_A and let φ be an endomorphism of G satisfying a Lipschitz condition. Then*

$$|g\varphi|_c \geq |g|_c \quad (38)$$

for every $g \in F_A$.

Proof. Suppose that there exists some $g \in F_A$ such that $|g\varphi|_c < |g|_c$. We may assume that g is cyclically reduced and $g\varphi = x^{-1}hx$ with h cyclically reduced. Hence $|h| < |g|$. For every $n \geq 1$, we have

$$\begin{aligned} (g^n|g^{n-1})_p^A - (g^n\varphi|g^{n-1}\varphi)_p^A &= \frac{1}{2}(d_A(p, g^n) + d_A(p, g^{n-1}) - d_A(g^n, g^{n-1}) - d_A(p, g^n\varphi) \\ &\quad - d_A(p, g^{n-1}\varphi) + d_A(g^n\varphi, g^{n-1}\varphi)) \\ &\geq \frac{1}{2}(2d_A(p, g^n) - 2d_A(g^n, g^{n-1}) - 2d_A(p, g^n\varphi) \\ &= d_A(p, g^n) - d_A(g, 1) - d_A(p, g^n\varphi) \\ &\geq d_A(1, g^n) - d_A(g, 1) - d_A(1, g^n\varphi) - 2d_A(p, 1) \\ &= |g^n| - |g| - |g^n\varphi| - 2|p| = n|g| - |g| - |x^{-1}h^n x| - 2|p| \\ &= n|g| - |g| - 2|x| - n|h| - 2|p| \geq n - |g| - 2|x| - 2|p|. \end{aligned}$$

Thus there is no constant $Q \in \mathbb{R}$ such that

$$(u\varphi|v\varphi)_p^A + Q \geq (u|v)_p^A$$

for all $u, v \in F_A$. In view of Proposition 3.4, this contradicts φ satisfying a Lipschitz condition. Therefore (38) holds as claimed. \square

We can give a complete solution in the case of free group automorphisms. We denote by $\text{Aut}(F_A)$ (respectively $\text{Inn}(F_A)$) the group of automorphisms (respectively inner automorphisms) of F_A . We say that $\varphi \in \text{Aut}(F_A)$ is a *permutation automorphism* (with respect to A) if $\varphi|_{\tilde{A}}$ is a permutation. Let $\text{Per}^A(F_A)$ denote the group of all permutation automorphisms of F_A with respect to A . Since

$$\lambda_x \varphi = \varphi \lambda_{x\varphi} \quad (39)$$

holds for all $\varphi \in \text{Aut}(F_A)$ and $x \in F_A$, and $\text{Inn}(F_A) \trianglelefteq \text{Aut}(F_A)$, $\text{Per}^A(F_A) \leq \text{Aut}(F_A)$, it follows easily that

$$\langle \text{Inn}(F_A) \cup \text{Per}^A(F_A) \rangle = \text{Per}^A(F_A) \text{Inn}(F_A). \quad (40)$$

Theorem 6.4 *Let $d \in V^A(p, \Gamma, T)$ be a visual metric on F_A and let $\varphi \in \text{Aut}(F_A)$. Then the following conditions are equivalent:*

- (i) φ satisfies a Lipschitz condition;
- (ii) $\varphi \in \langle \text{Inn}(F_A) \cup \text{Per}^A(F_A) \rangle = \text{Per}^A(F_A) \text{Inn}(F_A)$.

Proof. (i) \Rightarrow (ii). Since φ is an automorphism, it induces a natural bijection $\overline{\varphi}$ on the set C of conjugacy classes of F_A : if $[g]$ denotes the conjugacy class of g , we set $[g]\overline{\varphi} = [g\varphi]$.

Cyclic length extends naturally to conjugacy classes by

$$|[g]|_c = |g|_c.$$

We claim that

$$|[g]\overline{\varphi}|_c = |[g]|_c \quad (41)$$

holds for every $g \in F_A$. Indeed, if (41) fails, the fact that $\overline{\varphi}$ is a bijection and there are only finitely many conjugacy classes of a given cyclic length implies that $|[g]\overline{\varphi}|_c < |[g]|_c$ for some $g \in F_A$. Hence $|g\varphi|_c < |g|_c$, contradicting Lemma 6.3. Thus (41) holds. It follows that

$$|g\varphi|_c = |g|_c \quad (42)$$

for every $g \in F_A$.

Let $y \in G$ be such that

$$k = |\{a \in A : |a\varphi\lambda_y| = 1\}|$$

is maximum and let $\varphi' = \varphi\lambda_y$.

Suppose that there exists some $c \in A$ such that $|c\varphi'| > 1$. We may assume that $A = \{a_1, \dots, a_n\}$, $a_i\varphi' = b_i \in \tilde{A}$ for $i = 1, \dots, k$ and $c\varphi' = z^{-1}dz$ in reduced form. Since $|g\varphi'|_c = |g\varphi|_c$ for every $g \in F_A$, it follows from (42) that $d \in \tilde{A}$.

Suppose first that $k > 1$. Then there exist $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ such that $(a_1^{\varepsilon_1}ca_2^{\varepsilon_2})\varphi' = b_1^{\varepsilon_1}z^{-1}dz b_2^{\varepsilon_2}$ is a cyclically reduced word. Since $|a_1^{\varepsilon_1}ca_2^{\varepsilon_2}|_c = 3$ and

$$|(a_1^{\varepsilon_1}ca_2^{\varepsilon_2})\varphi'|_c = |b_1^{\varepsilon_1}z^{-1}dz b_2^{\varepsilon_2}|_c = 3 + 2|z| \geq 5,$$

this contradicts (42).

Thus $k = 1$. Let $\varepsilon \in \{1, -1\}$ be such that $(a_1^\varepsilon c)\varphi' = b_1^\varepsilon z^{-1}dz$ is a reduced word. Write $z = wb_1^m$ in reduced form with $|m|$ maximum. Then

$$(a_1^\varepsilon c)\varphi' = b_1^\varepsilon z^{-1}dz = b_1^{\varepsilon-m}w^{-1}dw b_1^m.$$

Similarly to the previous case, $w \neq 1$ implies

$$|(a_1^\varepsilon c)\varphi'|_c = 2|w| + 2 > 2 = |a_1^\varepsilon c|_c,$$

hence we may assume that $w = 1$. But then $z = b_1^m$ and

$$a_1\varphi\lambda_{yz^{-1}} = b_1^m b_1 b_1^{-m} = b_1, \quad c\varphi\lambda_{yz^{-1}} = c\varphi\lambda_y\lambda_{z^{-1}} = c\varphi'\lambda_{z^{-1}} = b_1^m z^{-1}dz b_1^{-m} = d,$$

contradicting the maximality of k .

We have thus reached a contradiction in any case following from $k < |A|$, hence $\varphi' \in \text{Per}^A(F_A)$ and $\varphi = \varphi'\lambda_{y^{-1}} \in \text{Per}^A(F_A)\text{Inn}(F_A)$. By (40), condition (ii) holds.

(ii) \Rightarrow (i). Permutation automorphisms are clearly isometries of (F_A, d_A) , and satisfying a Lipschitz condition is preserved under composition. Now the claim follows from Proposition 6.1 and Lemma 6.2. \square

We note also that Theorem 6.4 is far from covering the case of endomorphisms. For instance, it is easy to check that the monomorphism φ of F_A defined by $a\varphi = a^2$ for every $a \in A$ satisfies a Lipschitz condition for every visual metric of the form $d \in V^A(p, \gamma, T)$.

We discuss next what happens when we want to consider arbitrary finite generating sets. An important role will be played by $\epsilon^A \in \text{Per}^A(F_A)$ defined by $a\epsilon^A = a^{-1}$ ($a \in A$).

Theorem 6.5 *Let $\varphi \in \text{Aut}(F_A)$. Then the following conditions are equivalent:*

(i) φ satisfies a Lipschitz condition for every visual metric on F_A ;

(ii) $\varphi \in \text{Inn}(F_A)$ or $|A| \leq 1$.

Proof. (i) \Rightarrow (ii). In view of Proposition 6.1 and Theorem 6.4, it suffices to show that every $\varphi \in \text{Per}^A(F_A) \setminus \{1\}$ fails condition (i) when $|A| > 1$.

Suppose first that $a\varphi \notin \{a, a^{-1}\}$ for some $a \in A$. Let $b = a\varphi$ and let $A' = (A \setminus \{b, b^{-1}\}) \cup \{u\}$, where $u = ab$. It is immediate that A' is an alternative basis of F_A . Since $a\varphi = a^{-1}u$, we have $|a\varphi|_c > |a|_c$ in $F_{A'} = F_A$. Thus φ fails condition (42) in the proof of Theorem 6.4 when we consider a visual metric $d = \sigma_{p, \gamma}^{A'}$.

Thus we may assume that $a\varphi = a^{-1}$ for some $a \in A$. Suppose that $b\varphi = b$ for some $b \in A \setminus \{a\}$. Let $A' = (A \setminus \{b\}) \cup \{u\}$, where $u = ab$. Since $u\varphi = a^{-1}b = a^{-2}u$, we have $|u\varphi|_c > |a|_c$ in $F_{A'} = F_A$. Thus φ fails condition (42) in the proof of Theorem 6.4 when we consider a visual metric $d = \sigma_{p, \gamma}^{A'}$.

Finally, we assume that $\varphi = \epsilon^A$. Fix $b \in A \setminus \{a\}$ and let $A' = A \cup \{u, v\}$, where $u = a^2b$ and $v = a^3b$. Let $w = a^{-2}b^{-1}a^{-3}b^{-1}ab$. We claim that

$$d_{A'}(1, w^n) = 5n \tag{43}$$

for every $n \in \mathbb{N}$.

Indeed, since $w = a^{-2}v^{-1}u^{-1}v$, we have $d_{A'}(1, w) \leq 5$ and so $d_{A'}(1, w^n) \leq 5n$. Assume now that

$$w^n = y_1 x_1 y'_1 x'_1 y''_1 x''_1 y_2 x_2 y'_2 x'_2 y''_2 x''_2 \dots y_n x_n y'_n x'_n y''_n x''_n z$$

where the $x_i, x'_i, x''_i \in \widetilde{A'}$ produce the $3n$ occurrences of b/b^{-1} in the reduced form of w^n , and y_i, y'_i, y''_i, z are words on $\widetilde{A'}$. Then each y_i must equal a^{-2} after reduction in F_A . Since x_i starts by b^{-1} and x''_{i-1} ends by b (if $i > 1$), it follows easily that y_i must have at least two factors from $\widetilde{A'}$. Therefore

$$d_{A'}(1, w^n) \geq 3n + 2n = 5n$$

and so (43) holds.

On the other hand,

$$w^n \varphi = (w\varphi)^n = (a^2 b a^3 b a^{-1} b^{-1})^n = (u v a^{-1} b^{-1})^n$$

yields $d_{A'}(1, w^n \varphi) \leq 4n$. Hence

$$(w_n | w_n)_1^{A'} - (w_n \varphi | w_n \varphi)_1^{A'} = d_{A'}(1, w^n) - d_{A'}(1, w^n \varphi) \geq 5n - 4n = n$$

and so there is no constant $Q \in \mathbb{R}$ such that

$$(g\varphi | h\varphi)_1^{A'} + Q \geq (g | h)_1^{A'}$$

for all $g, h \in F_A$. By Proposition 3.4, φ cannot satisfy a Lipschitz condition with respect to a visual metric $\sigma_{1,\gamma}^{A'}$.

(ii) \Rightarrow (i). If φ is inner, we use Proposition 6.1. If $|A| \leq 1$, it is easy to see that φ is an isometry of $(F_A, d_{A'})$ for every finite generating set A' of F_A , therefore we may apply Lemma 6.2. \square

The reader may have been surprised by the use of a generating set which is not a basis in the proof of Theorem 6.5. The next two propositions show that we could have done it for $|A| > 2$, but not for $|A| = 2$:

Proposition 6.6 *Let $\varphi \in \text{Aut}(F_A)$. If $|A| > 2$, then the following conditions are equivalent:*

(i) φ satisfies a Lipschitz condition for every visual metric $d \in V^{A'}(p, \gamma, T)$ on F_A , where A' is a basis of F_A ;

(ii) $\varphi \in \text{Inn}(F_A)$.

Proof. By the proof of Theorem 6.5, it suffices to show that ϵ^A satisfies no Lipschitz condition for some visual metric $\sigma_{p,\gamma}^{A'}$ on F_A , where A' is a basis of F_A .

Fix distinct $a, b, c \in A$. Once again, let $A' = (A \setminus \{b\}) \cup \{u\}$, where $u = ab$. It is immediate that A' is an alternative basis of F_A . Since

$$(uc)\epsilon^A = (abc)\epsilon^A = a^{-1}b^{-1}c^{-1} = a^{-1}u^{-1}ac^{-1},$$

we have $|(uc)\epsilon^A|_c > |uc|_c$ in $F_{A'} = F_A$. Thus ϵ^A fails condition (42) in the proof of Theorem 6.4 when we consider a visual metric $d = \sigma_{p,\gamma}^{A'}$. Therefore ϵ^A satisfies no Lipschitz condition with respect to $\sigma_{p,\gamma}^{A'}$. \square

We fix a canonical basis $A = \{a, b\}$ for F_2 and write $\epsilon = \epsilon^A$. We note that, similarly to (40),

$$\langle \text{Inn}(F_2) \cup \{\epsilon\} \rangle = \langle \epsilon \rangle \text{Inn}(F_2). \quad (44)$$

Given a basis $\{u, v\}$ of F_2 , we denote by $\mu_{u,v}$ the automorphism of F_2 defined by $a \mapsto u$ and $b \mapsto v$.

Proposition 6.7 *Let $\varphi \in \text{Aut}(F_2)$. Then the following conditions are equivalent:*

- (i) φ satisfies a Lipschitz condition for every visual metric $d \in V^{A'}(p', \gamma', T)$ on F_2 , where A' is a basis of F_2 ;
- (ii) $\varphi \in \langle \text{Inn}(F_2) \cup \{\epsilon\} \rangle = \langle \epsilon \rangle \text{Inn}(F_2)$.

Proof. (i) \Rightarrow (ii). By the proof of Theorem 6.5.

(ii) \Rightarrow (i). Let $H = \langle \text{Inn}(F_2) \cup \{\epsilon\} \rangle$. It is well known that $\text{Aut}(F_2)$ is generated by the Nielsen transformations $N = \{\mu_{b,a}, \mu_{a^{-1},b}, \mu_{ab,b}\}$. It is easy to check that

$$\begin{aligned} \mu_{b,a}^{-1} \epsilon \mu_{b,a} &= \mu_{b,a} \mu_{b^{-1},a^{-1}} = \epsilon, \\ \mu_{a^{-1},b}^{-1} \epsilon \mu_{a^{-1},b} &= \mu_{a^{-1},b} \mu_{a,b^{-1}} = \epsilon, \\ \mu_{ab,b}^{-1} \epsilon \mu_{ab,b} &= \mu_{ab^{-1},b} \mu_{b^{-1}a^{-1},b^{-1}} = \mu_{b^{-1}a^{-1}b,b^{-1}} = \epsilon \lambda_b, \end{aligned}$$

thus $H \trianglelefteq \text{Aut}(F_2)$.

Let $d' = \sigma_{p',\gamma'}^{A'}$, where $A' = \{u, v\}$ is a basis of F_2 . Then $\mu = \mu_{u,v}$ is an automorphism of F_2 such that $A\mu = A'$. Thus, for all $g, h \in F_2$, we have

$$d_{A'}(g, h) = d_A(g\mu^{-1}, h\mu^{-1}). \quad (45)$$

We claim that

$$(g|h)_{p'}^{A'} = (g\mu^{-1}|h\mu^{-1})_{p'\mu^{-1}}^A \quad (46)$$

holds for all $g, h \in F_2$. Indeed, (45) yields

$$\begin{aligned} (g|h)_{p'}^{A'} &= \frac{1}{2}(d_{A'}(p', g) + d_{A'}(p', h) - d_{A'}(g, h)) \\ &= \frac{1}{2}(d_A(p'\mu^{-1}, g\mu^{-1}) + d_A(p'\mu^{-1}, h\mu^{-1}) - d_A(g\mu^{-1}, h\mu^{-1})) = (g\mu^{-1}|h\mu^{-1})_{p'\mu^{-1}}^A. \end{aligned}$$

Now (46) yields

$$(g\epsilon|h\epsilon)_{p'}^{A'} = (g\mu^{-1}\mu\epsilon|h\mu^{-1}\mu\epsilon)_{p'}^{A'} = (g\mu^{-1}\mu\epsilon\mu^{-1}|h\mu^{-1}\mu\epsilon\mu^{-1})_{p'\mu^{-1}}^A. \quad (47)$$

Since $H \trianglelefteq F_2$, we have $\mu\epsilon\mu^{-1} \in H$ and so by Theorem 6.4 $\mu\epsilon\mu^{-1}$ satisfies a Lipschitz condition with respect to any visual metric $d = \sigma_{p'\mu^{-1},\gamma'}^A$. By Proposition 3.4, there exists a constant $Q \in \mathbb{R}$ such that

$$(g\mu\epsilon\mu^{-1}|h\mu\epsilon\mu^{-1})_{p'\mu^{-1}}^A + Q \geq (g|h)_{p'\mu^{-1}}^A$$

holds for all $g, h \in G$. Together with (47) and (46), this yields

$$(g\epsilon|h\epsilon)_{p'}^{A'} + Q = (g\mu^{-1}\mu\epsilon\mu^{-1}|h\mu^{-1}\mu\epsilon\mu^{-1})_{p'\mu^{-1}}^A + Q \geq (g\mu^{-1}|h\mu^{-1})_{p'\mu^{-1}}^A = (g|h)_{p'}^{A'}.$$

By Proposition 3.4, ϵ satisfies a Lipschitz condition with respect to d . Now the general case follows from Proposition 6.1 and satisfying a Lipschitz condition being preserved by composition. \square

We can deduce from the proof of Proposition 6.7 a curious result proved by Kapovich, Levitt, Schupp and Shpilrain in 2007. We recall that $g \in F_A$ is *primitive* if it belongs to some basis of F_A . If $g = a_1 a_2 \dots a_n$ in reduced form ($a_i \in \tilde{A}$), the *reversal* of g (with respect to A) is defined as $g^R = a_n \dots a_2 a_1$.

Corollary 6.8 [10, Proposition 3.1] *The conjugacy class of a primitive word of F_2 is closed under reversal.*

Proof. Let u be a primitive word of F_2 in reduced form. Then there exists some $\mu_{u,v} \in \text{Aut}(F_2)$. Since $H \trianglelefteq F_2$, we have $\mu_{u,v} \epsilon \mu_{u,v}^{-1} \in H$. Indeed, it follows from the proof of Proposition 6.7 that

$$\mu_{u,v} \epsilon \mu_{u,v}^{-1} = \epsilon \lambda_x$$

for some $x \in F_2$. Hence (39) yields

$$(u^R)^{-1} = u \epsilon = a \mu_{u,v} \epsilon = a \epsilon \lambda_x \mu_{u,v} = a^{-1} \mu_{u,v} \lambda_x \mu_{u,v} = u^{-1} \lambda_x \mu_{u,v}$$

and so

$$u^R = u \lambda_x \mu_{u,v}.$$

Since a conjugate of a primitive word is itself primitive, we are done. \square

In particular, if u is a cyclically reduced primitive word, then u^R , being cyclically reduced, must be a cyclic conjugate of u .

Note that Corollary 6.8 does not hold for higher rank (take e.g. $u = abc$) or for nonprimitive elements of F_2 (take e.g. $u = aba^2b^2$).

7 Open problems

The main open problem left by this work relates to the possibility of removing quasiconvexity from condition (v) of Theorem 4.3.

Problem 7.1 *Does every uniformly continuous endomorphism of a hyperbolic group (with respect to a visual metric) satisfy a Hölder condition? If not, would it satisfy some other type of condition?*

It would be interesting to extend some of the results in Section 6 to hyperbolic, or at least virtually free groups:

Problem 7.2 *When does an automorphism of a hyperbolic (virtually free) group satisfy a Lipschitz condition?*

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References

- [1] V. Araújo and P. V. Silva, Geometric characterizations of virtually free groups, preprint, arXiv:1405.5400, 2014.
- [2] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wissenschaften, Volume 319, Springer, New York, 1999.
- [3] J. Dugundji, *Topology*, Allyn and Bacon, 1966.
- [4] A. H. Frink, Distance functions and the metrization problem, *Bull. Amer. Math. Soc.* 43 (1937), 133–142.
- [5] E. Ghys and P. de la Harpe (eds), *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhauser, Boston, 1990.
- [6] M. Gromov, Random walk in random groups, *Geom. Funct. Anal.* 13(1) (2003), 73–146.
- [7] I. Holopainen, U. Lang and A. Vähäkangas, Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces, *Math. Ann.* 339(1) (2007), 101–134.
- [8] I. Kapovich, A non-quasiconvexity embedding theorem for hyperbolic groups, *Math. Proc. Cambridge Philos. Soc.* 127(3) (1995), 461–486.
- [9] I. Kapovich and N. Benakli, Boundaries of hyperbolic groups, In: *Combinatorial and geometric group theory*, *Contemp. Math.* 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39–93.
- [10] I. Kapovich, G. Levitt, P. Schupp and V. Shpilrain, Translation equivalence in free groups, *Trans. Amer. Math. Soc.* 359(4) (2007), 1527–1546.
- [11] I. Kapovich and D. T. Wise, On the failure of the co-hopf property for subgroups of word-hyperbolic groups, *Isr. J. Math.* 122(1) (2001), 125–147.
- [12] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, 1977.
- [13] E. Rips and Z. Sela, Structure and rigidity in hyperbolic groups I, *Geom. Funct. Anal.* 4(3) (1994), 337–371.
- [14] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups II, *Geom. Funct. Anal.* 7(3) (1997), 561–593.
- [15] H. Short, Quasiconvexity and a theorem of Howson's, In: E. Ghys, A. Haefliger and A. Verjovsky (eds.), *Group theory from a geometrical viewpoint (Trieste, 1990)*, World Sci. Publishing, River Edge, NJ, 1991, pp. 168–176.
- [16] P. V. Silva, Fixed points of endomorphisms of virtually free groups, *Pacific J. Math.* 263(1) (2013), 207–240.
- [17] J. Väisälä, Gromov hyperbolic spaces, *Expositiones Math.* 23(3) (2005), 187–231.

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